Tutorial 1 2020/09/14 Math 2010 B

Outline of tutorial

1. Quick review
2. geometric applications of dot product
3. Cauchy - Schwartz inequ & Triangle inequ

Quick Review:

The study of vectors (linear algebra) gives us a systemtic way to manipulate certain geometric objects. e.g lines. via arithmetics.

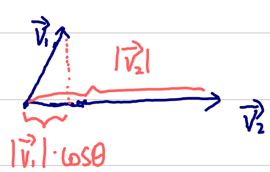
Geo. Obj	Alg obj
A line L w/ direction $\longrightarrow$	A vector $\vec{v}$
The length of L	$\Vert \vec{v} \Vert$
$L_1 \parallel L_2$	$\exists$ some scalars c_1, c_2 not all zero. s.t $\underset{L_1}{c_1\vec{v}_1} + \underset{L_2}{c_2\vec{v}_2} = 0$ (linear dependence) ($\exists\ \lambda$ s.t $\vec{v}_1 = \underset{\text{scalar}}{\lambda} \vec{v}_2$ or $\vec{v}_2 = \lambda \vec{v}_1$)
$L_1 \perp L_2$	$\vec{v}_1 \cdot \vec{v}_2 = 0$

One remark: dot product & cross product

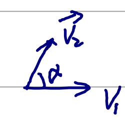
$\vec{v}_1$, $\vec{v}_2$

① $\vec{v}_1 \cdot \vec{v}_2 \overset{?}{=} \underline{|\vec{v}_1|} \cdot |\vec{v}_2| \cdot \underline{\cos\alpha} \in \mathbb{R}$

② $\underline{\vec{v}_1 \times \vec{v}_2} \overset{?}{=}$ vector



$\vec{V_1} \times \vec{V_2} \overset{?}{=} \underline{\text{vector}} \triangleq \vec{V_3}$

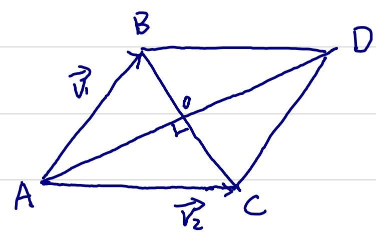


$\Rightarrow \vec{V_1} \perp \vec{V_3}$ & $\vec{V_2} \perp \vec{V_3}$ & $|\vec{V_3}| = |\vec{V_1}| \cdot |\vec{V_2}| \cdot \sin\alpha$

e.g. Show that diagonals of a rhombus are perpendicular.

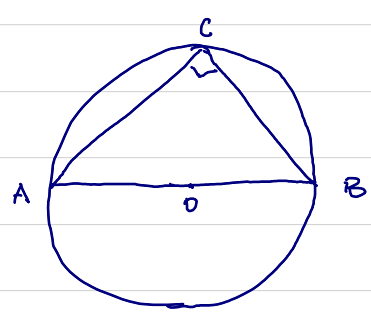
① Def of rhombus ?

② what do we need to show in terms of vectors ?



$(\vec{V_1} + \vec{V_2}) \cdot (\vec{V_1} - \vec{V_2}) = 0$ $\qquad \vec{V_1} + \vec{V_2} \overset{?}{=} \overrightarrow{AD}$ $\qquad \vec{V_1} - \vec{V_2} = \overrightarrow{CB}$

e.g 2. Show that $\angle ACB = 90°$



i.e need to show $\overrightarrow{AC} \cdot \overrightarrow{BC} = 0$

Cauchy - Schwarz Inequality

Let $\underset{(a_1, \dots a_n)}{\underset{\shortparallel}{\vec{a}}}, \vec{b} \in \mathbb{R}^n$. Then

$$|\underbrace{\vec{a} \cdot \vec{b}}_{\in \mathbb{R}}| \leq \|\vec{a}\| \|\vec{b}\|$$

More precisely: $\left| \sum_{i=1}^{n} a_i b_i \right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}$

& equality holds $\Leftrightarrow$ $\exists$ r. s $\in \mathbb{R}$ s.t $r\vec{a} + s\vec{b} = \vec{0}$

① lecture - proof By quadratic function

$n=2$ $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$ $\because$ $-1 \leq \cos\theta \leq 1$

② Alternative proof:

Fix $\vec{a}, \vec{b} \in \mathbb{R}^n$

Case (1) Either $\vec{a} = 0$ or $\vec{b} = 0$

can easily check $|\vec{a} \cdot \vec{b}| = 0 = |\vec{a}| |\vec{b}|$

& $\exists$ $r = 1$. $\underset{0}{\underset{\shortparallel}{s}} \in \mathbb{R}$. s.t $r\vec{a} + s \cdot \vec{b} = 0$ (w/ assume $a=0$)

Case (2) $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{0}$

Let $\vec{a}' = \dfrac{\vec{a}}{\|\vec{a}\|}$ and $b' = \dfrac{\vec{b}}{\|\vec{b}\|}$ $\Rightarrow$ $\|\vec{a}'\| = \|\vec{b}'\| = 1$

$$2(1 \pm \vec{a}' \cdot \vec{b}') \overset{\Delta}{=} \underset{1}{\|\vec{a}'\|^2} \pm 2\vec{a}' \cdot \vec{b}' + \underset{1}{\|\vec{b}'\|^2} = \|\vec{a}' + \vec{b}'\|^2 \geq 0$$

$$\Rightarrow \quad 1 - (\vec{a}' \cdot \vec{b}')^2 = \overset{\geq 0}{(1 + \vec{a}' \cdot \vec{b}')} \overset{\geq 0}{(1 - \vec{a}' \cdot \vec{b}')} \overset{=0}{\geq} 0 \quad \Rightarrow \quad |\vec{a}' \cdot \vec{b}'| \leq 1$$

$$\vec{a}' \cdot \vec{b}' = \frac{\vec{a} \cdot \vec{b}}{\|a\| \|b\|}$$

$\Leftrightarrow$ $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \|\vec{b}\|$

$$|\vec{a}\cdot\vec{b}\,\| = \|\vec{a}\|\|\vec{b}\|$$

$\Leftrightarrow$ Either $1+\vec{a}\cdot\vec{b}' = 0$ or $1-\vec{a}\cdot\vec{b}' = 0$

$\Leftrightarrow$ Either $\exists\ r, s \in \mathbb{R}\setminus\{0\}$, s.t $r\vec{a}+s\vec{b} = 0$.

App 1: Triangle Inequality ✓

App 2: Let $\vec{a}, \vec{b} \in \mathbb{R}^n \setminus \{0\}$

$$-1 \leq \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} \leq -1$$

$\therefore$ we can well define the angle between $\vec{a}, \vec{b}$ as

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|a\|\,\|b\|}\right) \in [0, \pi]$$